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## ***On the Connection between Binary Quartics and Elliptic Functions.***

BY E. STUDY.

The object of the following research is an application of the theory developed in the preceding paper to elliptic functions.

The binary quartic has played an important part in elliptic functions from the very beginnings of this theory, and the subject was early considered from the standpoint of the modern theory of invariants. A prominent result was Hermite's famous transformation;\* a pair of important formulæ has been communicated by Weierstrass in his lectures;† some no less important results were added by F. Klein,‡ who at the same time applied similar methods to hyperelliptic functions, and originated in this way a series of investigations of quite a new character; finally, we have to mention a dissertation of Burkhardt,§ who extended his considerations to some irrational covariants.

Our starting point is different from those of the authors mentioned, and in some respects more elementary. We compare the relations among the rational and irrational covariants of a quartic with the identities among the four  $\Theta$ -functions. Simple as this idea may be, nevertheless a new light is thrown by it upon the familiar formulæ, and at the same time a number of new results can be derived, which make the theory in question in a certain sense *complete*.

The method applied being so elementary, we need not dwell upon the details of the proofs; but we may lay some stress upon the fact that all our

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\* Crelle's Journal, v. 52. See also Clebsch, Binäre Formen, §62, and Weber, Elliptische Functionen, I, §4.

† Biermann, Problema mechanica (Diss., Berlin, 1865).

‡ Hyperelliptische Sigmafunctionen, Math. Ann., vol. 27, §12, p. 454.

§ Beziehungen zwischen der Invariantentheorie und der Theorie der algebraischen Integrale und ihrer Umkehrungen, München, 1887.

results are obtained by means of *actual calculations*, referring to the *general* case of a binary quartic (not merely to a canonical form), and that no use whatever is made of what is called the method of indeterminate coefficients. Neither do we take results from one theory and apply them to the other. Such proceedings are excellent means of investigation when new fields are opened, but in a more advanced state of science they are hardly satisfactory. The interest of the connection between quartics and elliptic functions is, besides, not lessened by the fact that either theory permits the deduction of its theorems by means of its own.

The  $\Theta$ -functions used in this research are not quite identical with Weierstrass'  $\Theta$ -functions. We refer to the author's paper, "On the Addition Theorems of Jacobi and Weierstrass" (*Am. Journ.*, Vol. XVI, p. 156), the results of which are supposed to be known to the reader.\*

The preceding paper is simply quoted as "Quartic."

### §1. *The Four $\Theta$ -Functions and the Linear Forms $(r_\kappa x)$ .*

In order to compare the theory of binary quartics with the theory of elliptic functions, we identify the irrational invariants  $\sqrt{e_\mu - e_\nu}$ , etc., with the quantities  $\sqrt{e_\mu - e_\nu}$ , etc., belonging to the latter theory. Then of course our rational invariants  $g_2, g_3, G$  coincide with the quantities  $g_2, g_3, G$  of Weierstrass. Now comparing the linear relations among the forms  $(r_\kappa x)$  (Quartic No. 70) with the identities among the squares of the functions  $\Theta_\kappa u$ , we see that we are enabled to explain a set of square roots by the formulæ

$$\sqrt{(r_\mu r_\nu)} \cong \Theta_\lambda(0) = \Theta_\lambda, \text{ etc.}, \quad (1)$$

$$\sqrt{\sqrt{G}} = \sqrt[4]{G} \cong \Theta'(0) = \Theta' \quad (2)$$

(where  $\Theta' = \Theta_\lambda(0) \Theta_\mu(0) \Theta_\nu(0) = \Theta_\lambda \Theta_\mu \Theta_\nu$ ), and

$$\begin{aligned} \sqrt{(r_0 x)} &\cong \Theta u, & \sqrt{(r_\lambda x)} &\cong \Theta_\lambda u, \\ \sqrt{(r_\mu x)} &\cong \Theta_\mu u, & \sqrt{(r_\nu x)} &\cong \Theta_\nu u. \end{aligned} \quad (3)$$

Considering  $u$  as a variable,† we have established a one-to-two correspon-

\* I use this opportunity to correct a misprint: On p. 156 write  $\Re\left(\frac{\omega_2}{\omega_1 i}\right)$  and  $\Re\left(\frac{\omega_3}{\omega_1 i}\right)$  instead of  $\Re\left(\frac{\omega_2}{\omega_1}\right)$  and  $\Re\left(\frac{\omega_3}{\omega_1}\right)$ .

† It must not be overlooked that after the establishment of equations of the form (3), the binary variables  $x_1$  and  $x_2$ , belonging to a definite system of coordinates, are no longer independent quantities, although the ratio  $x_1 : x_2$  continues to be arbitrary. Namely, when the coefficients of the forms  $(r_\kappa x)$  are

dence between the values of  $u$ , situated within the parallelogram of periods, and the points  $x$  of the binary domain. When  $u$  varies throughout the parallelogram, the point  $x$  varies through the binary domain, obtaining each situation twice; *vice versa*, to every point belong two values  $\pm u$ . Points of exception are only the points  $x = r_0, r_\lambda, r_\mu, r_\nu$ , to which correspond only single values of  $u$ , namely, half periods. And if  $u$  varies throughout the plane representing the complex values of  $u$ , the forms  $(r_\kappa x)$  defining the corresponding point of the binary domain are reproduced, excepting an exponential factor, common to all of them. Proceeding from  $u$  to  $u' = u \pm 2\omega_\lambda$ ,  $(r_\kappa x)$  is changed into  $(r_\kappa x') = e^{4\eta_\lambda \omega_\lambda} \cdot e^{\pm 4\eta_\lambda u} \cdot (r_\kappa x)$ . In the same way the square roots  $\sqrt[r_\kappa x]{}$  are reproduced with an exponential factor common to all of them, when we surpass the boundaries of the *double* parallelogram of periods. Passing from  $u$  to  $u' = u \pm 4\omega_\lambda$ ,  $\sqrt[r_\kappa x]{}$  is changed into  $\sqrt[r_\kappa x']{=} e^{8\eta_\lambda \omega_\lambda} \cdot e^{\pm 4\eta_\lambda u} \cdot \sqrt[r_\kappa x]{}$ , as we could partly have anticipated by considering the two-leaved Riemann surface defined by the branchpoints  $r_0, r_\lambda, r_\mu, r_\nu$ .

Now we may operate principally upon the surface just mentioned, introducing the new irrationality  $\sqrt[r]{f}$  by the formula

$$\sqrt[r]{f} = -2\sqrt[r_0 x]{} \sqrt[r_\lambda x]{} \sqrt[r_\mu x]{} \sqrt[r_\nu x]{}. \quad (4)$$

Considering  $f$  and its covariants as functions of  $u$ , we draw immediately the conclusions:

1). *f and all rational covariants of f, and certain irrational covariants besides, are one-valued functions of the argument u, which are reproduced with an exponential factor, when u is augmented by a period.*

2). *All quotients of such covariants, the degree of which in x is zero (viz. all absolute covariants of f), are elliptic functions of u.*

Of course the formulæ (3), the basis of our further considerations, cannot claim any novelty, as far as their general form is concerned.\* We may call attention, however, to the circumstance that we have made special suppositions in the definition of the linear forms  $(r_\kappa x)$  on the one side and in the definition of

known, the periods  $2\omega$  and  $2\omega'$  can be determined as functions of these coefficients. Substituting, then, a definite value of  $u$  in the functions  $\Theta_\kappa u$ , not only the ratio of  $x_1$  and  $x_2$  is determined, but also the absolute values of these quantities. Starting therefore from *given* values of  $x_1$  and  $x_2$ , we have to add a factor of proportionality, say  $\rho(x)$ . It does not seem necessary, however, to carry this factor visibly through the whole investigation; we may unite it with the variables  $x_1$  and  $x_2$  and denote the products  $\rho x_1$  and  $\rho x_2$  again by  $x_1$  and  $x_2$ .—See the footnotes in §2 and §6.

\* Compare Weber, *Elliptische Functionen und algebraische Zahlen* (Braunschweig, 1893), §36.

our  $\Theta$ -functions on the other. It is entirely due to the exact parallelism in the two definitions that we are enabled to derive from these formulæ *simple* consequences; an apparently slight change of notation would imply considerable algebraical complications.

## §2. The Elliptic Functions $\varphi u$ , $\varphi' u$ .

We now proceed to establish the expressions of the elliptic functions  $\varphi u$ ,  $\varphi' u$  in terms of covariants of  $f$ . Considering the distribution of the argument  $u$  on the Riemann surface defined by  $\sqrt{f}$ , we cannot expect to find the values of these functions in the domain of  $\sqrt{f}$  and the rational covariants of  $f$ ; whereas, as we shall see, the contrary holds in the case of the functions of the double argument  $\varphi(2u)$ ,  $\varphi'(2u)$ .

Expressing  $\varphi(u)$ , etc., in terms of the  $\Theta$ -functions, we find, after a short calculation,

$$\begin{aligned} \frac{(r_\nu r_\lambda)(r_\lambda r_\mu) \cdot (r_\lambda x) + (r_\lambda r_\mu)(r_\mu r_\nu) \cdot (r_\mu x) + (r_\mu r_\nu)(r_\nu r_\lambda) \cdot (r_\nu x)}{3(r_0 x)} &= \\ = \frac{(tr_0)^5(tx)}{2\sqrt{G} \cdot (r_0 x)} &= \frac{(ar_0)^2(ax)^2}{2(r_0 x)^2} \cong \varphi u, \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{-(r_\nu r_\lambda)(r_\lambda r_\mu) \cdot (r_0 x) + (r_\lambda r_\mu)(r_\mu r_\nu) \cdot (r_\nu x) - (r_\mu r_\nu)(r_\nu r_\lambda) \cdot (r_\mu x)}{3(r_\lambda x)} &= \\ = \frac{(tr_\lambda)^5(tx)}{2\sqrt{G} \cdot (r_\lambda x)} &= \frac{(ar_\lambda)^2(ax)^2}{2(r_\lambda x)^2} \cong \varphi(u \pm \omega_\lambda), \end{aligned}$$

$$\begin{aligned} -2(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) \cdot \frac{\sqrt{(r_\lambda x)} \sqrt{(r_\mu x)} \sqrt{(r_\nu x)}}{\sqrt{(r_0 x)} \sqrt{(r_0 x)} \sqrt{(r_0 x)}} &= \frac{\sqrt{2} (tr_0)^3(tx)^3}{\sqrt{(r_0 x)} \sqrt{(r_0 x)} \sqrt{(r_0 x)}} = \\ = \sqrt[4]{G} \cdot \frac{\sqrt{f}}{(r_0 x)^2} &= \frac{(ar_0)^3(ax) \cdot \sqrt{f}}{(r_0 x)^3} \cong \varphi' u, \end{aligned} \quad (6)$$

$$\begin{aligned} -2(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) \cdot \frac{\sqrt{(r_0 x)} \sqrt{(r_\nu x)} \sqrt{(r_\mu x)}}{\sqrt{(r_\lambda x)} \sqrt{(r_\lambda x)} \sqrt{(r_\lambda x)}} &= \frac{\sqrt{2} (tr_\lambda)^3(tx)^3}{\sqrt{(r_\lambda x)} \sqrt{(r_\lambda x)} \sqrt{(r_\lambda x)}} = \\ = \sqrt[4]{G} \cdot \frac{\sqrt{f}}{(r_\lambda x)^2} &= \frac{(ar_\lambda)^3(ax) \cdot \sqrt{f}}{(r_\lambda x)^3} \cong \varphi'(u \pm \omega_\lambda), \end{aligned} \quad (6)$$

$$\frac{3(tr_0)^4(tx)^2}{\sqrt[4]{G} \cdot (r_0 x)^2} = 2\sqrt[4]{G} \cdot \frac{(ar_0)(ax)^3}{(r_0 x)^3} \cong \varphi'' u, \quad (7)$$

$$\frac{3(tr_\lambda)^4(tx)^2}{\sqrt[4]{G} \cdot (r_\lambda x)^2} = 2\sqrt[4]{G} \cdot \frac{(ar_\lambda)(ax)^3}{(r_\lambda x)^3} \cong \varphi''(u \pm \omega_\lambda),$$

etc.

To facilitate the understanding of the significance of these formulæ we call to mind the following relations among the elliptic functions on the right side of (5), (6) and (7):

$$\begin{aligned}\wp(u \pm \omega_\lambda) &= \frac{(e_\mu - e_\lambda)(e_\nu - e_\lambda)}{\wp u - e_\lambda} + e_\lambda, \\ \wp'(u \pm \omega_\lambda) &= -(e_\mu - e_\lambda)(e_\nu - e_\lambda) \cdot \frac{\wp' u}{(\wp u - e_\lambda)^2}, \\ \wp' u \cdot \wp'(u \pm \omega_\lambda) \cdot \wp'(u \pm \omega_\mu) \cdot \wp'(u \pm \omega_\nu) &= 16 G, \\ \frac{1}{\wp' u} + \frac{1}{\wp'(u \pm \omega_\lambda)} + \frac{1}{\wp'(u \pm \omega_\mu)} + \frac{1}{\wp'(u \pm \omega_\nu)} &= 0, \\ \wp'' u &= 6\wp^2 u - \frac{1}{2} g_2.\end{aligned}$$

### §3. *The $\Theta$ -Functions of the Double Argument.*

Passing to the double argument, we obtain a very much simpler set of formulæ:

$$\sqrt{f} \cong -\Theta' \cdot \Theta(2u),^* \quad (8)$$

$$\begin{aligned}l &\cong -\Theta_\lambda \cdot \Theta_\lambda(2u), \\ m &\cong -\Theta_\mu \cdot \Theta_\mu(2u), \\ n &\cong -\Theta_\nu \cdot \Theta_\nu(2u);\end{aligned} \quad (9)$$

consequently we have

$$t \cong 2\Theta'^\circ \cdot \Theta_\lambda(2u) \Theta_\mu(2u) \Theta_\nu(2u), \quad (10)$$

$$t\sqrt{f} \cong -2\Theta'^\circ \cdot \Theta(4u), \quad (11)$$

and

$$-\frac{h}{f} \cong \wp(2u), \quad (12)$$

$$\frac{t}{f\sqrt{f}} \cong \wp'(2u), \quad (13)$$

$$-3 \frac{(f, t)}{f^2} = 6 \frac{h^2}{f^2} - \frac{1}{2} g_2 \cong \wp''(2u), \quad (14)$$

$$12 \frac{(f, (f, t))}{f^2 \sqrt{f}} = -12 \frac{h \cdot t}{f^2 \sqrt{f}} \cong \wp'''(2u), \text{ etc.} \quad (15)$$

\* From (3) and (8) follows:

$$\frac{\sqrt{(r_\kappa x)}}{\sqrt[4]{f}} \cong \frac{\Theta_\kappa u}{\sqrt[4]{-\Theta' \cdot \Theta(2u)}} \quad (\kappa = 0, \lambda, \mu, \nu).$$

These formulæ are able to replace the formulæ (3) in most respects. They are independent of the above supposition concerning the values of the binary variables.

The formulæ (12) and (13) are immediately evident; the higher differential quotients of  $\wp(2u)$  are easily obtained by means of Gordan's series (Clebsch, Binäre Formen, §8; Gordan's Vorlesungen, v. II, §7), see §6, No. 47.

The corresponding expressions of  $\wp(2u + \omega_\lambda)$ , etc., contain, of course, the irrationality  $e_\lambda$ :

$$-\frac{f}{l^2} + e_\lambda \cong \wp(2u + \omega_\lambda), \quad (12b)$$

$$-\frac{\sqrt{f} \cdot t}{(e_\mu - e_\lambda)(e_\nu - e_\lambda) \cdot l^4} = -2(e_\mu - e_\nu) \cdot \frac{\sqrt{f} \cdot mn}{l^3} \cong \wp'(2u + \omega_\lambda). \quad (13b)$$

The formula (12) is due to Hermite; it contains the transformation of the quartic  $f$  to Weierstrass' canonical form (Quartic No. 14, 19) by rational operations. It shows that such sets of eight values of  $u$  as  $\pm u, \pm u + \omega_\lambda, \pm u + \omega_\mu, \pm u + \omega_\nu$  correspond to the vanishing points of the forms  $zf + \lambda h$ ; the values of  $u$ , corresponding to a given form of our pencil, are therefore the roots of the transcendental equation

$$\wp(2u) = \frac{z}{\lambda}.$$

Especially to the vanishing points of  $f$  itself ( $\lambda = 0$ ) correspond, as we have seen already, the demi-periods; whereas the primitive quarters of a period, the roots of the equation

$$(\wp(2u) - e_\lambda)(\wp(2u) - e_\mu)(\wp(2u) - e_\nu) = 0$$

correspond to the vanishing points of the sextic  $t$  (No. 10, 13). The vanishing points of  $\Phi = \frac{1}{2}(t, t)_2$  are the roots of the equation

$$\begin{aligned} 0 &= \frac{4}{3}g_2 \cdot \wp'(2u) \cdot [\wp(2u + \omega_\lambda) + \wp(2u + \omega_\mu) + \wp(2u + \omega_\nu)] = \\ &= 4g_2 [g_2 \cdot \wp(2u)^3 + 3g_3 \cdot \wp(2u) + \frac{1}{12}g_2^2] = \\ &= [2g_2 \cdot \wp(2u) + 3g_3 + \frac{4}{3}\sqrt{-3}\sqrt{G}] [2g_2 \cdot \wp(2u) + 3g_3 - \frac{4}{3}\sqrt{-3}\sqrt{G}] \end{aligned}$$

(Quartic No. 13, 48), and the vanishing points of  $\Psi = (t, (t, t)_2)_1$  are the roots of the equation

$$\begin{aligned} 0 &= 2g_3 \cdot \wp(2u)^3 + \frac{g_2^2}{3} \cdot \wp(2u)^2 + \frac{g_2 g_3}{2} \cdot \wp(2u) - \left( \frac{g_2^3}{108} - \frac{g_3^2}{2} \right) = \\ &= [2e_\lambda \cdot \wp(2u) - \frac{g_2}{3} + 2e_\lambda^2] [2e_\mu \cdot \wp(2u) - \frac{g_2}{3} + 2e_\mu^2] [2e_\nu \cdot \wp(2u) - \frac{g_2}{3} + 2e_\nu^2] \end{aligned}$$

(Quartic No. 37, 43). Finally we may consider the group of eight points defined by the equations  $\varphi u = 0$ ,  $\varphi(u + \omega_\lambda) = 0$ , etc. It corresponds to the one form of our pencil, the linear factors of which are proportional to the linear forms appearing in the numerator of the equations (5). This is the quartic

$$g_2^2 f - 16g_3 \cdot h,$$

as we see by means of the expression of  $\varphi(2u)$  in terms of  $\varphi u$ , or also by direct calculation, starting from the formulæ (5).

The formulæ (13)–(15), and the expressions of  $\varphi(2u) - e_\lambda$  to be derived from (8) and (9), have been communicated by Burkhardt in his dissertation. But he fails to give the formulæ (8)–(11); and his formulæ as well as his demonstrations contain a number of, as it seems to me, superfluous complications. Some of his results have been found independently by Harkness and Morley (A Treatise on the Theory of Functions, New York, 1893, §203).

#### §4. *Formulæ with Two Arguments $u$ , $v$ .*

The results contained in §3 are capable of an important generalization. Writing in No. (3)  $y$  and  $v$  instead of  $x$  and  $u$ , and considering the  $\Theta$ -functions of  $u$  and  $v$  at the same time, we obtain immediately the following remarkable set of formulæ :

$$\begin{aligned} (xy) &\cong -\Theta(u+v)\Theta(u-v), \\ (lx)(ly) &\cong -\Theta_\lambda(u+v)\Theta_\lambda(u-v), \\ (mx)(my) &\cong -\Theta_\mu(u+v)\Theta_\mu(u-v), \\ (nx)(ny) &\cong -\Theta_\nu(u+v)\Theta_\nu(u-v). \end{aligned} \tag{16}$$

Hence we derive

$$\begin{aligned} (tx)^3(ty)^3 &\cong 2\Theta'^4 \cdot \Theta_\lambda(u+v)\Theta_\lambda(u-v)\Theta_\mu(u+v)\Theta_\mu(u-v)\Theta_\nu(u+v)\Theta_\nu(u-v), \\ 2(xy) \cdot (tx)^3(ty)^3 &\cong -\Theta'^6 \cdot \Theta(2u+2v)\Theta(2u-2v), \end{aligned} \tag{17}$$

and

$$\begin{aligned} -\frac{\sqrt{(r_0x)}\sqrt{(r_\lambda x)} \cdot \sqrt{(r_\mu y)}\sqrt{(r_\nu y)}}{(xy)} \mp \frac{\sqrt{(r_0y)}\sqrt{(r_\lambda y)} \cdot \sqrt{(r_\mu x)}\sqrt{(r_\nu x)}}{(xy)} &\cong \\ \cong \Theta_\mu\Theta_\nu \cdot \frac{\Theta_\lambda(u \pm v)}{\Theta(u \pm v)} &= \frac{\mathcal{G}_\lambda(u \pm v)}{\mathcal{G}(u \pm v)} = \sqrt{\varphi(u \pm v) - e_\lambda}, \end{aligned} \tag{18}$$

$$\underline{\frac{(ly)^2 \sqrt{(ax)^4} \mp (lx)^2 \sqrt{(ay)^4}}{(xy)^2}} \cong \frac{\wp'(u \pm v)}{\sqrt{e_v - e_\lambda} \sqrt{e_\lambda - e_\mu}} \cdot \frac{\sqrt{\wp(u \mp v) - e_\lambda}}{\sqrt{\wp(u \pm v) - e_\lambda}}, \quad (19)$$

$$\underline{\frac{(ax)^2(ay)^2 \mp \sqrt{(ax)^4} \sqrt{(ay)^4}}{2(xy)^2}} \cong \wp(u \pm v), \quad (20)$$

$$\underline{\frac{(ay)^3(ax) \sqrt{(ax)^4} \mp (ax)^3(ay) \sqrt{(ay)^4}}{(xy)^3}} \cong \wp'(u \pm v). \quad (21)$$

Including in the “system of  $f$ ” the irrationality  $\sqrt{f}$ , we draw the conclusion :

*All covariants of the quartic  $f$  containing the two sets of variables  $x$  and  $y$  in the degree zero, are rational functions of  $\wp(u+v)$ ,  $\wp(u-v)$ ,  $\wp'(u+v)$ ,  $\wp'(u-v)$ , and vice versa.*

Namely, the said covariants are quotients of the integral covariants  $(xy)$ ,  $\sqrt{f_x}$ ,  $\sqrt{f_y}$ , and of the polars of the forms  $f$ ,  $h$ ,  $t$  containing  $x$  and  $y$ . But multiplying such a polar with a properly chosen power of  $(xy)$ , we obtain an integral function of  $(ax)^4$ ,  $(ax)^3(ay)$ ,  $(ax)^2(ay)^2$ ,  $(ax)(ay)^3$ ,  $(ay)^4$ . Now considering the formulæ (12), (13), (20), (21), and paying attention to the circumstance that  $\wp(2u)$ ,  $\wp'(2u)$ ,  $\wp(2v)$ ,  $\wp'(2v)$  are rational functions of  $\wp(u+v)$ , etc.,\* we obtain the above theorem. The inverse statement is immediately evident.

There are, among the absolute covariants in question, a number of quite interesting expressions. We mention the following examples :

$$\underline{\frac{(hx)^2(hy)^2}{(xy)^2}} \cong -\{\wp(u+v)\wp(u-v) + \frac{1}{12}g_2\}, \quad (22)$$

$$\begin{aligned} \underline{\frac{(ax)^3(ay) \cdot (hx)(hy)^3 - (ay)^3(ax) \cdot (hy)(hx)^3}{(xy)^4}} &= \frac{1}{2} \underline{\frac{(ax)^4(hy)^4 - (ay)^4(hx)^4}{(xy)^4}} = \\ &= \underline{\frac{(tx)^3(ty)^3}{(xy)^3}} \cong -\frac{1}{2}\wp'(u+v)\wp'(u-v), \end{aligned} \quad (23)$$

$$\underline{\frac{(hy)^3(hx) \cdot \sqrt{(ax)^4} \pm (hx)^3(hy) \cdot \sqrt{(ay)^4}}{(xy)^3}} \cong \wp(u \pm v)\wp'(u \mp v), \quad (24)$$

$$\underline{\frac{(ty)^4(tx)^2 \cdot \sqrt{(ax)^4} \pm (tx)^4(ty)^2 \cdot \sqrt{(ay)^4}}{(xy)^4}} \cong \frac{1}{3}\wp'(u \pm v)\wp''(u \mp v), \quad (25)$$

$$\begin{aligned} &\underline{\frac{(ax)^3(ay) \cdot (hx)(hy)^3 + (ay)^3(ax) \cdot (hy)(hx)^3}{(xy)^4}} = \\ &= \frac{1}{(xy)^4} \cdot \{(ax)^4 \cdot (hy)^4 + (ay)^4 \cdot (hx)^4 - \frac{1}{2}g_2 \cdot (xy)^3 \cdot (ax)^2(ay)^2 - \frac{3}{2}g_3(xy)^4\} \quad (26) \\ &\cong -2\{\wp(u+v)\wp(u-v) \cdot [\wp(u+v) + \wp(u-v)] + \frac{1}{4}g_3\}, \end{aligned}$$

\* See Schwarz, Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen (Göttingen, 1882), Art. 12.

$$\frac{(hx)^4 \cdot (hy)^4 - g_3 \cdot (xy)^2 \cdot (ax)^2 \cdot (ay)^2}{(xy)^4} = \left\{ \frac{(hx)^2 (hy)^2 - 1}{(xy)^2} (xy)^2 \cdot g_2 \right\}^2 \cong \{ \wp(u+v) \wp(u-v) + \frac{1}{4} g_2 \}^2, \quad (27)$$

$$\begin{aligned} \frac{(ty)^5 (tx) \cdot (ax)^4 - (tx)^5 (ty) \cdot (ay)^4}{(xy)^5} &= \\ &= \frac{1}{(xy)} \cdot \left\{ -8(hx)^4 \cdot (hy)^4 + \frac{2}{3} g_2 \cdot (ax)^4 \cdot (ay)^4 - \right. \\ &\quad \left. - 6 g_2 \cdot (xy)^2 \cdot (hx)^2 (hy)^2 + 9 g_3 \cdot (xy)^2 \cdot (ax)^2 (ay)^2 \right\} \quad (28) \\ &\cong -\frac{2}{3} \wp''(u+v) \wp''(u-v) + \frac{2}{3} g_2 \cdot \wp(u+v) \wp(u-v) \\ &\quad + g_3 \{ \wp(u+v) + \wp(u-v) \} + \frac{1}{18} g_2^2, \end{aligned}$$

$$\begin{aligned} \frac{(ty)^5 (tx) \cdot (ax)^4 + (tx)^5 (ty) \cdot (ay)^4}{(xy)^5} &= \frac{(ax)^2 (ay)^2}{(xy)^2} \cdot 2 \frac{(tx)^3 (ty)^3}{(xy)^3} \cong \quad (29) \\ &\cong -\{ \wp(u+v) + \wp(u-v) \} \cdot \wp'(u+v) \wp'(u-v). \end{aligned}$$

Generally, whenever a power of  $(xy)$  is the sole denominator, the absolute covariant is an integral function of  $\wp(u+v)$ ,  $\wp(u-v)$ ,  $\wp'(u+v)$ ,  $\wp'(u-v)$ . In other respects the properties of a given absolute covariant are determined by the distribution of the radicals  $\sqrt{f_x}$ ,  $\sqrt{f_y}$ .

The expressions on the left in (20) and (21) were first brought into connection with the elliptic functions  $\wp$  and  $\wp'$  by Weierstrass;\* and his formulæ have been reproduced, with some additional remarks, by F. Klein and others.†

These authors, however, do not seem to have grasped the full content of our formulæ (20) and (21). Namely, instead of considering a pair of arguments  $u+v$  and  $u-v$ , the authors quoted use only one, our  $u-v$  (which they denote by  $-u$ ). Thus they consider what we may term the *general distribution of the argument u* upon the Riemann surface defined by  $\sqrt{f_x}$  ( $u$  having the value zero at an arbitrary point—our point  $v$ —of the surface), but they do not show the relation between this general distribution and the ordinary distribution (where  $u$  vanishes at a branchpoint of the surface), consisting in the coexistence of the formulæ (20), (21) with the formulæ (3), (8), etc. We venture to consider it as an essential improvement that our theory puts the connection between the two distributions into evidence, and thus attributes to all the formulæ a more exact meaning.

\* Biermann, l. c.

† F. Klein, *Math. Ann.*, v. 27, §12, 13, p. 454–461.

## §5. Addition Theorems.

The preceding results permit us, of course, to pursue the parallelism between certain parts of the theory of elliptic functions and the theory of the binary quartic as far as we like. Especially the addition-theorems are easily brought now into a projective form. Take, for instance, three arguments  $u, v, w$ , connected by the relation

$$u + v + w \equiv 0 \pmod{\tilde{\omega}},$$

and denote the corresponding points by  $x, y, z$ ; then the relations hold

$$\Sigma (yz) \cdot (ty)^3(tz)^3 \cdot \sqrt[4]{(ay)^4} \sqrt[4]{(az)^4} \cdot (tx)^6 = 0,$$

and

$$\Omega_{yz} = \Omega_{zx} = \Omega_{xy},$$

where

$$\Omega_{yz} = \frac{(ty)^6 \cdot (az)^4 \cdot \sqrt[4]{(az)^4} - (tz)^6 \cdot (ay)^4 \sqrt[4]{(ay)^4}}{(yz) \cdot (ty)^3(tz)^3 \cdot \sqrt[4]{(ay)^4} \sqrt[4]{(az)^4}},$$

and when especially

$$u + v + w \equiv 0 \pmod{2\tilde{\omega}},$$

we have

$$\Sigma (yz) \cdot (r_0y)(r_0z) \cdot \sqrt[4]{(ax)^4} = 0,$$

or

$$\Sigma (yz) \cdot \sqrt[4]{(r_0y)} \cdot \sqrt[4]{(r_0z)} \cdot \sqrt[4]{(r_\lambda x)} \cdot \sqrt[4]{(r_\mu x)} \cdot \sqrt[4]{(r_\nu x)} = 0,$$

and

$$\Xi_{yz} = \Xi_{zx} = \Xi_{xy},$$

where

$$\Xi_{yz} = \frac{(r_0z)^2 \cdot \sqrt[4]{(ay)^4} - (r_0y)^2 \cdot \sqrt[4]{(az)^4}}{(yz) \cdot (r_0y)(r_0z)}.$$

As to the addition-theorems of the  $\Theta$ -functions, we evidently have in Quartic, §3, No. 26, a correlate to the sixteen addition-theorems of Weierstrass, belonging to our first family (I). (See *Am. Journal*, Vol. XVI, pp. 160, 161.) In the same way the nine families of the second type (II) have simple correlates, whereas to the six families of the third type (III) corresponds no equally simple result. We do not insist upon the general case, but we may point out the algebraic transformation corresponding to the transformation

$$\begin{aligned} u &= \frac{u_1 + v_1}{2}, & u_1 &= u + v, \\ v &= \frac{u_1 - v_1}{2}, & v_1 &= u - v. \end{aligned} \tag{30}$$

Denoting the binary variables corresponding to  $u_1$  and  $v_1$  by means of our formulæ (3) by  $\xi$  and  $\eta$  (whereas  $x$  and  $y$  correspond to  $u$  and  $v$ ), we find

$$\left. \begin{aligned} (xy) &= -\sqrt{r_0\xi} \sqrt{r_0\eta}, \\ (lx)(ly) &= -\sqrt{r_\lambda\xi} \sqrt{r_\lambda\eta}, \\ (mx)(my) &= -\sqrt{r_\mu\xi} \sqrt{r_\mu\eta}, \\ (nx)(ny) &= -\sqrt{r_\nu\xi} \sqrt{r_\nu\eta}, \end{aligned} \right\} \quad (31)$$

$$\left. \begin{aligned} \sqrt{(ax)^4 \cdot \sqrt{(ay)^4}} &= -\sqrt[4]{G} \cdot (\xi\eta), \\ (lx)^2 \cdot (ly)^2 &= -\sqrt{e_\mu - e_\nu} \cdot (l\xi)(l\eta), \\ (mx)^2 \cdot (my)^2 &= -\sqrt{e_\nu - e_\lambda} \cdot (m\xi)(m\eta), \\ (nx)^2 \cdot (ny)^2 &= -\sqrt{e_\lambda - e_\mu} \cdot (n\xi)(n\eta), \end{aligned} \right\} \quad (32)$$

$$\left. \begin{aligned} 2(ax)^4 \cdot (hy)^4 &= -\sqrt{G} \cdot \{(a\xi)^2(a\eta)^2 + \sqrt{(a\xi)^4} \cdot \sqrt{(a\eta)^4}\}, \\ 2(hx)^4 \cdot (ay)^4 &= -\sqrt{G} \cdot \{(a\xi)^2(a\eta)^2 - \sqrt{(a\xi)^4} \cdot \sqrt{(a\eta)^4}\}, \end{aligned} \right\} \quad (33)$$

$$\left. \begin{aligned} (tx)^6 \cdot (ay)^4 \cdot \sqrt{(ay)^4} &= \sqrt{G} \cdot \sqrt[4]{G} \cdot \{(a\eta)^3(a\xi) \cdot \sqrt{(a\xi)^4} - (a\xi)^3(a\eta) \cdot \sqrt{(a\eta)^4}\}, \\ (ty)^6 \cdot (ax)^4 \cdot \sqrt{(ax)^4} &= \sqrt{G} \cdot \sqrt[4]{G} \cdot \{(a\eta)^3(a\xi) \cdot \sqrt{(a\xi)^4} + (a\xi)^3(a\eta) \cdot \sqrt{(a\eta)^4}\}, \end{aligned} \right\} \quad (34)$$

$$\left. \begin{aligned} (hx)^4 \cdot (hy)^4 &= -\sqrt{G} \cdot \{(h\xi)^2(h\eta)^2 + \frac{1}{12}g_2 \cdot (\xi\eta)^2\}, \\ (hx)^4 \cdot (hy)^4 + \frac{1}{12}g_2 \cdot (ax)^4 \cdot (ay)^4 &= -\sqrt{G} \cdot (h\xi)^2(h\eta)^2, \end{aligned} \right\} \quad (35)$$

$$(tx)^6 \cdot (ty)^6 = 2\sqrt{G} \cdot \sqrt[4]{G} \cdot (t\xi)^3(t\eta)^3, \quad (36)$$

etc.

Putting  $v_1 = 0$ , that is to say,

$$2u = u_1, \quad (37)$$

we obtain the following special formulæ, expressing the *duplication of the argument*:

$$\left. \begin{aligned} \sqrt{(ax)^4} &= -\sqrt[8]{G} \cdot \sqrt{r_0\xi}, \\ (lx)^2 &= -\sqrt[4]{e_\mu - e_\nu} \cdot \sqrt{r_\lambda\xi}, \\ (mx)^2 &= -\sqrt[4]{e_\nu - e_\lambda} \cdot \sqrt{r_\mu\xi}, \\ (nx)^2 &= -\sqrt[4]{e_\lambda - e_\mu} \cdot \sqrt{r_\nu\xi}, \end{aligned} \right\} \quad (38)$$

$$(hx)^4 = -\frac{1}{\sqrt[4]{G}} \cdot (tr_0)^5(t\xi) = \sqrt[4]{G} \cdot \sqrt{-\{(hr_0)^2(h\xi)^2 + \frac{1}{12}g_2 \cdot (r_0\xi)^2\}}, \quad (39)$$

$$(tx)^6 = 2\sqrt{G} \sqrt[8]{G} \cdot \sqrt{r_\lambda\xi} \sqrt{r_\mu\xi} \sqrt{r_\nu\xi} = -\sqrt[4]{G} \sqrt[8]{G} \sqrt{2} (tr_0)^3(t\xi)^3, \quad (40)$$

$$(tx)^6 \cdot \sqrt{(ax)^4} = \sqrt{G} \cdot \sqrt[4]{G} \cdot \sqrt{(a\xi)^4}, \text{ etc.} \quad (41)$$

The transformations derived here from the theory of elliptic functions offer

themselves naturally from the mere algebraic standpoint too. We are led to the formulæ (31) by a comparison of the identity

$$(xy)^2 + [(lx)(ly)]^2 + [(mx)(my)]^2 + [(nx)(ny)]^2 = 0$$

with the identity

$$(r_0\xi)(r_0\eta) + (r_\lambda\xi)(r_\lambda\eta) + (r_\mu\xi)(r_\mu\eta) + (r_\nu\xi)(r_\nu\eta) = 0.$$

(See Quartic No. 24, 28, 71.)

Finally, we mention a curious relation referring to four arguments, the sum of which is a period. Supposing

$$u_0 + u_1 + u_2 + u_3 \equiv 0 \pmod{2\bar{\omega}}, \quad (42)$$

and denoting the  $x$  corresponding to the value  $u_\kappa$  of  $u$  by  $x_\kappa$ , we have

$$\begin{aligned} & (x_3x_3)(x_3x_1)(x_1x_2) \cdot \sqrt[4]{(ax_0)^4} - (x_3x_0)(x_0x_2)(x_2x_3) \cdot \sqrt[4]{(ax_1)^4} \\ & + (x_0x_1)(x_1x_3)(x_3x_0) \cdot \sqrt[4]{(ax_2)^4} - (x_1x_2)(x_2x_0)(x_0x_1) \cdot \sqrt[4]{(ax_3)^4} = 0. \end{aligned} \quad (43)$$

Namely, under the said condition the determinant of the four functions  $\Theta$  occurring in (8) and (9) vanishes ; developing this determinant, we obtain the theorem.

Replacing the condition (42) by the ampler condition

$$u_0 + u_1 + u_2 + u_3 \equiv 0 \pmod{\bar{\omega}}, \quad (44)$$

and denoting, for sake of shortness, the polar  $(tx_i)^3(tx_\kappa)^3$  by  $t_{i\kappa}$ , we have further

$$\begin{aligned} & (x_2x_3)(x_3x_1)(x_1x_2) \cdot t_{23} \cdot t_{31} \cdot t_{12} \cdot t_{00} \cdot \sqrt[4]{(ax_0)^4} \\ & - (x_3x_0)(x_0x_2)(x_2x_3) \cdot t_{30} \cdot t_{02} \cdot t_{23} \cdot t_{11} \cdot \sqrt[4]{(ax_1)^4} \\ & + (x_0x_1)(x_1x_3)(x_3x_0) \cdot t_{01} \cdot t_{13} \cdot t_{30} \cdot t_{22} \cdot \sqrt[4]{(ax_2)^4} \\ & - (x_1x_2)(x_2x_0)(x_0x_1) \cdot t_{12} \cdot t_{20} \cdot t_{01} \cdot t_{33} \cdot \sqrt[4]{(ax_3)^4} = 0. \end{aligned} \quad (45)$$

The last addition-theorem is obtained by developing the vanishing determinant

$$|h_0^2, h_1f_1, f_2^2, t_3\sqrt[4]{f_3}|.$$

The two addition-theorems (43) and (45) are transformed into each other by means of the transformation defined by (37)–(41). Namely, from these formulæ, or directly from (17) and (16), follows

$$2(xx') \cdot (tx)^3(tx')^3 = \sqrt{G} \cdot \sqrt[4]{G} \cdot (\xi\xi'), \quad (46)$$

$x$ ,  $x'$  and  $\xi$ ,  $\xi'$  corresponding to one another by means of (37)–(41). By their substitution (45) and (44) are reduced to (43) and (42).

Specializing our formulæ by means of the supposition  $u_0 = 0$ , we obtain again the addition-theorems communicated on p. 225.

### §6. *The Elementary Integrals.*

When in the formula (46)  $x$  and  $x'$  are brought close together by means of the supposition  $u' = u + du$ , we obtain, paying regard to (16) and (8),

$$-\frac{(xx')}{\sqrt[4]{f_x f_{x'}}} = -\frac{1}{2} \frac{(\xi \xi')}{\sqrt[4]{f_\xi f_{\xi'}}} \cong du = \frac{1}{2} du_1. \quad (47)$$

Replacing here  $x'$  and  $\xi'$  by  $x + dx$  and  $\xi + d\xi$ , we obtain immediately the following expression of the integral of the first kind  $u$ :

$$\int_x^{r_0} \frac{(zdz)}{\sqrt{f_z}} = \frac{1}{2} \int_\xi^{r_0} \frac{(zdz)}{\sqrt{f_z}} \cong u. \quad (48)$$

The path of integration, of course, has to be chosen so that when  $z$  passes from  $r_0$  to  $x$ ,  $u$  varies from 0 to  $u$ , and not merely to a congruent value  $u + 2\bar{\omega}$ .

\* Herewith the factor of proportionality, mentioned in the footnote on p. 217, is determined.

Replacing the formulæ (8) in which, as we have said already, only  $u$ , but not  $x$ , can be considered as an independent argument, by

$$\sqrt{\rho(x) \cdot (r_\kappa x)} \cong \Theta_\kappa(u) \quad (\kappa = 0, \lambda, \mu, \nu),$$

and considering here the binary variables as independent quantities, we have defined in this way a homogeneous function  $\rho(x)$  of the degree  $-1$ . Instead of (8) we obtain now

$$\rho^2(x) \cdot \sqrt{f_x} \cong -\Theta' \cdot \Theta(2u),$$

whereas (48) does not change its form.

Consequently we have

$$\rho(x) = \frac{1}{\sqrt[4]{f_x}} \cdot \sqrt{\Theta' \cdot \Theta \left\{ 2 \int_{r_0}^x \frac{(zdz)}{\sqrt{f_z}} \right\}}.$$

Adding this factor to every pair of binary variables we transform all our formulæ into identities. The binary variables may now be considered as independent quantities. But only their ratio continues to enter into our formulæ; hence when  $u$  is given, the formulæ fail to determine the binary variables completely, as the simpler formulæ used in the text actually do.

In a similar way the elementary integrals of the second and third kind are calculated. We find

$$\begin{aligned} & \frac{1}{2\sqrt{G}} \int_x^y \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{(tr_\kappa)^5(tz)}{(r_\kappa z)} \cong - \int_u^v dw \varphi(u + \omega_\kappa) = \\ & = \frac{\Theta'_\kappa v}{\Theta'_\kappa v} - \frac{\Theta'_\kappa u}{\Theta'_\kappa u} = \zeta_\kappa v - \zeta_\kappa u \quad (\kappa = 0, \lambda, \mu, \nu; \quad \omega_0 = 0; \quad \Theta_0 = \Theta, \quad \zeta_0 = \zeta) \end{aligned} \quad (49)$$

$$-2 \int_x^y \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{(hz)^4}{(az)^4} = \frac{1}{\sqrt{G}} \int_\xi^v \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{(tr_0)^5(tz)}{(r_0 z)} \cong -2 \int_u^v dw \varphi(2w) = \zeta(2v) - \zeta(2u) \quad (50)$$

$$-2 \int_x^y \frac{(zdz)}{\sqrt{f_z}} \cdot \left\{ \frac{f_z}{l_z^2} - e_\lambda \right\} \cong \zeta_\lambda(2v) - \zeta_\lambda(2u), \quad (50b)$$

and further,

$$\int_{x_1}^{x_2} \frac{(xdx)}{\sqrt{f_x}} \cdot \frac{(ax)^2(ay)^2}{(xy)^2} \cong \left\{ \begin{array}{l} \zeta(u_2 + v) + \zeta(u_2 - v) - \\ - \zeta(u_1 + v) - \zeta(u_1 - v), \end{array} \right. \quad (51)$$

$$\int_{x_1}^{x_2} \frac{(xdx)}{\sqrt{f_x}} \cdot \frac{(al)^2(ax)^2 \cdot (ly)^2 - (ax)^2(ay)^2}{[(lx)(ly)]^2} \cong \left\{ \begin{array}{l} \zeta_\lambda(u_2 + v) + \zeta_\lambda(u_2 - v) - \\ - \zeta_\lambda(u_1 + v) - \zeta_\lambda(u_1 - v). \end{array} \right. \quad (51b)$$

Integrating once more, this time with respect to the parameter  $v$ , we obtain

$$\begin{aligned} & - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{(xdx)}{\sqrt{f_x}} \cdot \frac{(ydy)}{\sqrt{f_y}} \cdot \frac{(ax)^2(ay)^2}{(xy)^2} \\ & \cong \lg \frac{\Theta(u_1 + v_1)\Theta(u_2 + v_2)}{\Theta(u_1 - v_1)\Theta(u_2 - v_2)} \cdot \frac{\Theta(u_1 - v_2)\Theta(u_2 - v_1)}{\Theta(u_1 + v_2)\Theta(u_2 + v_1)}; \end{aligned} \quad (52)$$

the corresponding formula (52b) may be omitted. Further we have

$$-\sqrt{f_y} \cdot \int_{x_1}^{x_2} \frac{(xdx)}{(xy)^2} \cong \left\{ \begin{array}{l} \zeta(u_2 + v) - \zeta(u_2 - v) \\ - \zeta(u_1 + v) + \zeta(u_1 - v) \end{array} \right\} = \frac{\partial}{\partial v} \lg \frac{\varphi u_2 - \varphi v}{\varphi u_1 - \varphi v}, \quad (53)$$

$$-\sqrt{f_y} \cdot \int_{x_1}^{x_2} \frac{(xdx)}{[(lx)(ly)]^2} \cong \left\{ \begin{array}{l} \zeta_\lambda(u_2 + v) - \zeta_\lambda(u_2 - v) \\ - \zeta_\lambda(u_1 + v) + \zeta_\lambda(u_1 - v) \end{array} \right\} = \frac{\partial}{\partial v} \lg \frac{\varphi u_2 - \varphi(v + \omega_\lambda)}{\varphi u_1 - \varphi(v + \omega_\lambda)}. \quad (53b)$$

Integrating once more with respect to the parameter, we obtain

$$\begin{aligned} & \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{(xdx)(ydy)}{(xy)^2} = \lg \frac{(x_1 y_1)(x_2 y_2)}{(x_1 y_2)(x_2 y_1)} \cong - \int_{u_1}^{u_2} \int_{v_1}^{v_2} du dv [\varphi(u + v) - \varphi(u - v)] \\ & = -\frac{1}{2} \lg \frac{\varphi(u_1 + v_1) - \varphi(u_1 - v_1)}{\varphi(u_1 + v_2) - \varphi(u_1 - v_2)} \cdot \frac{\varphi(u_2 + v_2) - \varphi(u_2 - v_2)}{\varphi(u_2 + v_1) - \varphi(u_2 - v_1)} \\ & = \lg \frac{\varphi u_1 - \varphi v_1}{\varphi u_1 - \varphi v_2} \cdot \frac{\varphi u_2 - \varphi v_2}{\varphi u_2 - \varphi v_1} = \lg \frac{\Theta(u_1 + v_1)\Theta(u_1 - v_1)}{\Theta(u_1 + v_2)\Theta(u_1 - v_2)} \cdot \frac{\Theta(u_2 + v_2)\Theta(u_2 - v_2)}{\Theta(u_2 + v_1)\Theta(u_2 - v_1)}, \end{aligned} \quad (54)$$

and similarly

$$\begin{aligned} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{(xdx)(ydy)}{[(lx)(ly)]^2} &= \lg \frac{(lx_1)(ly_1) \cdot (lx_2)(ly_2)}{(lx_1)(ly_2) \cdot (lx_2)(ly_1)} \cong \\ &\cong \lg \frac{\Theta_\lambda(u_1 + v_1) \Theta_\lambda(u_1 - v_1)}{\Theta_\lambda(u_1 + v_2) \Theta_\lambda(u_1 - v_2)} \cdot \frac{\Theta_\lambda(u_2 + v_2) \Theta_\lambda(u_2 - v_2)}{\Theta_\lambda(u_2 + v_1) \Theta_\lambda(u_2 - v_1)}. \end{aligned} \quad (54b)$$

Hence comparing (51) and (53), we obtain

$$\int_{x_1}^{x_2} \frac{(xdx)}{\sqrt{f_x}} \cdot \frac{(ax)^2(ay)^2 \mp \sqrt{f_x} \sqrt{f_y}}{2(xy)^2} \cong \zeta(u_2 \pm v) - \zeta(u_1 \pm v), \quad (55)$$

and a similar equation is obtained by comparison of (51b) and (53b); and comparing (52) and (54), we have

$$\begin{aligned} \mp \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{(xdx)}{\sqrt{f_x}} \cdot \frac{(ydy)}{\sqrt{f_y}} \cdot \frac{(ax)^2(ay)^2 \mp \sqrt{f_x} \sqrt{f_y}}{2(xy)^2} &\cong \\ &\cong \lg \frac{\Theta(u_1 \pm v_1) \Theta(u_2 \pm v_2)}{\Theta(u_1 \pm v_2) \Theta(u_2 \pm v_1)}, \text{ etc.} \end{aligned} \quad (56)$$

We denote the double integral on the left, adopting the *lower* sign, by  $Q \left[ \begin{smallmatrix} x_2 y_2 \\ x_1 y_1 \end{smallmatrix} \right]$ , and write (56) thus:

$$\begin{aligned} Q \left[ \begin{smallmatrix} x_2 y_2 \\ x_1 y_1 \end{smallmatrix} \right] &\cong \lg \frac{\Theta(u_1 - v_1) \Theta(u_2 - v_2)}{\Theta(u_1 - v_2) \Theta(u_2 - v_1)}, \\ Q \left[ \begin{smallmatrix} x_2 \bar{y}_2 \\ x_1 \bar{y}_1 \end{smallmatrix} \right] &\cong \lg \frac{\Theta(u_1 + v_1) \Theta(u_2 + v_2)}{\Theta(u_1 + v_2) \Theta(u_2 + v_1)}, \end{aligned} \quad (56)$$

the replacement of  $y$  by  $\bar{y}$  indicating that the system of values  $y, \sqrt{f_y}$  is replaced by the *conjugate* system of values  $y, -\sqrt{f_y}$ .

An easy specialization of our formulæ furnishes now expressions of the functions  $\zeta_\kappa u$ , etc., themselves in terms of covariant integrals: *We have simply to introduce conjugate limits in the above formulæ.* Thus we obtain

$$Z_\kappa(x) = -\frac{1}{4\sqrt{G}} \int_x^{\bar{x}} \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{(tr_\kappa)^5(tz)}{(r_\kappa z)} \cong \zeta_\kappa u \quad (x = 0, \lambda, \mu, \nu), \quad (57)$$

$$3(x) = \int_x^{\bar{x}} \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{h_z}{f_z} \cong \zeta(2u), \quad (58)$$

$$3_\lambda(x) = \int_x^{\bar{x}} \frac{(zdz)}{\sqrt{f_z}} \cdot \left\{ \frac{f_z}{l_z^2} - e_\lambda \right\} \cong \zeta_\lambda(2u), \quad (58b)$$

$$\begin{aligned} -\frac{1}{4} \left\{ \int_x^{\bar{x}} \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{(ay)^2(az)^2}{(yz)^2} + \int_y^{\bar{y}} \frac{(zdz)}{\sqrt{f_z}} \cdot \frac{(ax)^2(az)^2}{(xz)^2} \right\} &= Z(x, y) \cong \zeta(u + v); \\ Z(x, \bar{y}) &\cong \zeta(u - v), \end{aligned} \quad (59)$$

etc. The mutual relations of the functions  $Z_k(x)$  and  $Z(x, y)$  introduced here are evident. As to  $Z(x)$ , we notice that  $Z(x, y)$  remains finite and definite when  $y$  coincides with  $x$ .  $Z(x, x) = \lim_{y \rightarrow x} Z(x, y)$  represents, however, an infinity of different analytic functions of  $x$ , corresponding to the different suppositions  $v = u + 2\omega$ . One of them is our  $Z(x)$ ; it corresponds to the simplest assumption  $v = u$ , that is to say, to the supposition that in the integrals contained in (59) not only the limits, but also the paths of integration are brought close together.

A similar remark holds with respect to the function  $Q \left[ \begin{smallmatrix} x_2 & y_2 \\ x_1 & y_1 \end{smallmatrix} \right]$  defined in No. (56). We denote the function of  $x_1, x_2$ , which results when  $v_1$  coincides with  $u_1$  and  $v_2$  with  $u_2$ , by  $Q \left( \begin{smallmatrix} x_2 \\ x_1 \end{smallmatrix} \right)$ . Then we have

$$Q \left( \begin{smallmatrix} y \\ x \end{smallmatrix} \right) \cong \lg \frac{\Theta(2u)\Theta(2v)}{\Theta^2(u+v)};$$

on the other hand, according to (8) and (16),

$$\frac{(xy)}{\sqrt[4]{f_x f_y}} \cong \frac{\Theta(u+v)\Theta(u-v)}{\Theta' \cdot \sqrt{\Theta(2u)\Theta(2v)}},$$

consequently

$$\begin{aligned} \frac{(xy)}{\sqrt[4]{f_x f_y}} \cdot e^{\frac{1}{2}Q \left( \begin{smallmatrix} y \\ x \end{smallmatrix} \right)} &\cong \frac{\Theta(u-v)}{\Theta'} = \sigma(u-v), \\ \sqrt{-1} \cdot \frac{(xy)}{\sqrt[4]{f_x f_y}} \cdot e^{\frac{1}{2}Q \left( \begin{smallmatrix} y \\ x \end{smallmatrix} \right)} &\cong \frac{\Theta(u+v)}{\Theta'} = \sigma(u+v); \end{aligned} \quad (60)$$

further, denoting the products  $\sqrt{(r_0 x)}\sqrt{(r_\lambda x)}$  and  $\sqrt{(r_\mu x)}\sqrt{(r_\nu x)}$  by  $\sqrt{\psi_x}$  and  $\sqrt{\chi_x}$ , on account of (18),

$$\begin{aligned} -\frac{\sqrt{\psi_x}\sqrt{\chi_y} + \sqrt{\psi_y}\sqrt{\chi_x}}{\sqrt[4]{f_x f_y}} \cdot e^{\frac{1}{2}Q \left( \begin{smallmatrix} y \\ x \end{smallmatrix} \right)} &\cong \frac{\Theta_\lambda(u-v)}{\Theta_\lambda} = \sigma_\lambda(u-v), \\ -\sqrt{-1} \frac{\sqrt{\psi_x}\sqrt{\chi_y} - \sqrt{\psi_y}\sqrt{\chi_x}}{\sqrt[4]{f_x f_y}} \cdot e^{\frac{1}{2}Q \left( \begin{smallmatrix} y \\ x \end{smallmatrix} \right)} &\cong \frac{\Theta_\lambda(u+v)}{\Theta_\lambda} = \sigma_\lambda(u+v). \end{aligned} \quad (60b)$$

The greater part of the preceding results is not new. Especially the formulæ (56), (60) and (60b) have been studied by F. Klein and H. Burkhardt (Math. Annalen, v. 27, 32), after Klein, paying regard mainly to what we have called the general distribution of the argument, had introduced the first formulæ

in (60) and (60b) as definitions of the  $\mathcal{G}$ -functions. There are, however, some difficulties connected with these expressions which seem to make further investigations desirable, notwithstanding the valuable researches of Burkhardt. The function  $Q\left(\frac{y}{x}\right)$  is not so easy to deal with as one might wish on account of its central position in the theory, apparently for the reason of its definition as the limit of a double integral. We have not been able to get rid of these complications without introducing others. Only one expression, of  $\mathcal{G}(2u)$ , may therefore be mentioned, which is remarkable for the part the covariant  $\Phi$  (Quartic No. 36) plays in it:

$$\mathcal{G}(2u) \cong \frac{\sqrt{f_x}}{\sqrt{-\frac{T}{2}}} \cdot e^{-6 \int_{r_0}^x \frac{(ydy)}{\sqrt{f_y}} \int_y^{\bar{y}} (zdz)} \cdot \frac{\sqrt{f_z} \cdot \Phi_z}{T_z^2}. \quad (61)$$

$T$  is the sextic covariant,  $= t$ .

### §7. Additional Remarks concerning the Transformation of some Integrals.

The results hitherto developed establish the connection between two series of analytical expressions: The homogeneous functions on the left side of our formulæ refer to the general case of the binary quartic, the functions on the right to Weierstrass' canonical form, and to the special shape the theory of elliptic functions assumes when this form is made the basis of the calculations. It is easy, of course, to specialize our considerations in such a way that any given canonical form of the quartic appears on the left. Thus we might investigate the connection among different shapes of the theory, based upon the consideration of a binary quartic, without losing the general points of view furnished by the theory of invariants. It is not our intention to dwell upon these details; we may call attention, however, to a certain canonical form of the quartic which has not yet been considered by previous authors, as far as we know.

Its properties are, in a certain respect, opposite to those of Weierstrass' canonical form, and it seems to offer some advantage when not only the argument  $u$ , but also the periods  $\omega, \omega'$  are considered as variables. In this case in Weierstrass' canonical form three of the vanishing points of the quartic undergo a change of situation, whereas the last one, the point infinity, alone rests fixed. But we might just as well keep three points in fixed positions and permit the

last one alone to vary. The fixed points may be any three arbitrarily chosen points of the binary domain; for instance, the three cube roots of unity, or the points 0, 1,  $\infty$  ("Riemann's canonical form," see Klein's *Modulfunctionen*, I, p. 25). Instead of specializing the fixed points in this way, we prefer to consider them as the vanishing points of a given but entirely *arbitrary cubic*, and to identify the corresponding linear factors with the irrational covariants  $(\lambda x)$ ,  $(\mu x)$ ,  $(\nu x)$  of this cubic. *Thus the canonical form in question is characterized by the fact that the sum of three linear factors of the quartic vanishes identically.*

We have two different modes of transforming a given quartic into this canonical form which can evidently be reached only through irrational operations. The first consists simply in replacing the products  $(r_\mu r_\nu) \cdot (r_\lambda x)$ ,  $(r_\nu r_\lambda) \cdot (r_\mu x)$ ,  $(r_\lambda r_\mu) \cdot (r_\nu x)$ , by  $(\lambda x)$ ,  $(\mu x)$ ,  $(\nu x)$  respectively; it is a linear transformation. The second way is shown by the theory of the octahedron, developed in "Quartic," §8. We shall confine ourselves to the second case, considering the transformation of the elliptic integral of the first kind.

Writing the formulæ (59)–(61) on p. 192 once more, with a second pair of corresponding variables  $\eta$ ,  $y$  (instead of  $\xi$ ,  $x$ ), we find, after a short calculation,

$$\sqrt{-r} \cdot (\xi \eta) \cong 4\sqrt{-R} \cdot (xy) \cdot (tx)^3 (ty)^3; \quad (62)$$

consequently, when  $\eta$  and  $y$  nearly coincide with  $\xi$  and  $x$ ,

$$\sqrt{-r} \cdot \frac{(\xi d\xi)}{\sqrt{p}} \cong 4\sqrt{-R} \cdot (xdx). \quad (63)$$

On the other hand, denoting the sum  $e_\lambda(\lambda\xi) + e_\mu(\mu\xi) + e_\nu(\nu\xi)$  by  $3(\rho\xi)$ , we have

$$\sqrt{-r} \cdot (\rho\xi) \cong \sqrt{-R} \cdot (ax)^4. \quad (64)$$

Combining the last pair of formulæ and choosing  $r_0$  as the lower limit of the integral referring to the variable  $x$ , we obtain the transformation in question:

$$\int_p^\xi \frac{(\xi d\xi)}{2\sqrt{p} \cdot \sqrt{(\rho\xi)}} \cong 2 \int_{r_0}^x \frac{(xdx)}{\sqrt{f}}. \quad (65)$$

The interest of the formula consists mainly in the decomposition of the elliptic radical into a cubic and a linear factor, *the first of which is entirely independent of the quartic  $f$ .*

The equality (65) is another expression of the one-to-four correspondence between  $\xi$  and  $x$  established by the formulæ (59)–(61) on p. 192. Consequently the two corresponding periods of integrals in (65) must be *equal*, and the invariants

of  $f$  must be equal to the corresponding invariants of the quartic  $4p \cdot (\rho\xi)$ . This is actually the case, as a simple calculation shows. *Thus we see that our principle of transference, by which the covariants of a cubic are transformed into the covariants of the octahedron, represents a peculiar interpretation of the duplication of the argument in the theory of elliptic functions.*

Although it does not exactly belong to our present topic, we may mention that the same principle of transference furnishes still some other remarkable transformations of integrals.

We have, for instance, on account of (63):

$$\begin{aligned} \sqrt[6]{-r} \int \frac{(\xi d\xi)}{p^{\frac{1}{3}}} &\cong 4\sqrt[6]{-R} \int \frac{(xdx)}{T^{\frac{1}{3}}}, \\ \sqrt{-r} \int \frac{(\xi d\xi)}{q^{\frac{1}{3}}} &\cong 4\sqrt{-R} \int (xdx) \cdot \frac{T}{\Psi^{\frac{1}{3}}}. \end{aligned}$$

By means of these formulæ the integrals on the right are recognized as elliptic integrals, belonging to the equianharmonic case (the case in which complex multiplication by a cube root of unity takes place). The reduction of the first integral is known; it is due to Brioschi (*Sulla equazione dell' ottaedro*, Trans. della R. Acc. dei Lincei, ser. III, v. III, p. 233).

Another transformation worth mentioning is the following one:

$$(-r)^{\frac{1}{3}} \int \frac{(\xi d\xi)(\rho\xi)}{\sqrt{p} \sqrt{q}} \cong 4(-R)^{\frac{1}{3}} \int (xdx) \cdot \frac{(ax)^4}{\sqrt{\Psi}}.$$

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